Electromagnetic Scattering from an Infinite Circular Metallic Cylinder Coated by an Elliptic Dielectric One

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Abstract—**In this paper the scattering from an infinite circular, perfectly conducting cylinder, coated by an elliptic dielectric one, is considered. The electromagnetic field is expressed in terms of both circular and elliptical cylindrical wave functions, which are connected with one another by well-known expansion formulas. In the special case of small $h = ka/2$ (a is the interfocal distance of the elliptic dielectric and k its wavenumber), exact, closed-form expressions of the form $S(h) = S(0)[1 + gh^2 + O(h^4)]$ are obtained for the scattered field and the various scattering cross sections of the problem. Both polarizations are considered for normal incidence. Graphical results for various values of the parameters are given.**

I. Introduction

The SCATTERING of an electromagnetic plane wave by an infinite circular perfectly conducting cyclinder coated by an elliptic dielectric one, is considered in the present paper. The reasons for considering analytical and closed-form solutions to problems of this kind are referred in [1]–[4]. Such solutions have practical importance, parallely to their mathematical interest. Our results are useful for the detection of metallic bodies embedded in dielectrics. The analogous problem, with the circular and elliptic boundaries interchanged, was solved elsewhere [5]. The elliptical-circular combination of this work may enhance or decrease the various scattering cross sections, in comparison with the corresponding ones of the problem of two coaxial circular cylinders, a result which also appeared in [5] and in problems of scattering from eccentrically coated infinite metallic cylinders and spheres [3], [4].

The solution of the problem under consideration is not made in steps totally analogous to those of the corresponding one in [5], but it presents much more complexities and lengthy manipulations, because the external boundary of the dielectric coating is elliptic now (circular in [5]). This fact raises enough more difficulties in the satisfaction of the boundary conditions and the evaluation of the sets of linear nonhomogeneous equations for the expansion coefficients of the field, as well as in the manipulation of these coefficients, as will become evident in what follows.

Manuscript received August 22, 1991; revised April 1, 1992.
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IEEE Log Number 9202894.

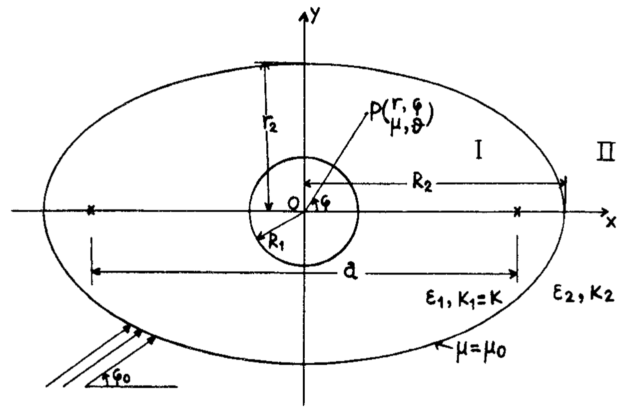

Fig. 1. Geometry of the scatterer.

II. Solution of the Problem for the E-Wave Polarization

The geometry of the problem is shown in Fig. 1. The two cylinders have the same z axis. We designate by $\mu = \mu_0$ the elliptical boundary and take notice of the following basic relations:

$$R_2 = \frac{a}{2}\cosh\mu_0, \quad r_2 = \frac{a}{2}\sinh\mu_0,$$
$$h = ka/2, \quad h\cosh\mu_0 = kR_2,$$
$$h_2 = k_2a/2, \quad h_2\cosh\mu_0 = k_2R_2,$$
$$a^2 = 4(R_2^2 - r_2^2), \quad \mu_0 = \tanh^{-1}(r_2/R_2).$$

The dielectric constant and the wavenumber are $\epsilon_1, k_1(= k)$ in region I and $\epsilon_2(= \epsilon_0), k_2(= k_0)$ in region II (free space). The magnetic permeability of both regions is that of free space.

We consider first the E-wave polarization. The incident plane wave impinging normally on the z axis has the form [6], [7]:

$$E_z^{\text{inc}} = \sqrt{8\pi}\sum_{m=0}^{\infty} j^{-m}\left\{\left[\frac{Se_m(h_2,\cos\varphi_0)}{M_m^e(h_2)}\right] \cdot Se_m(h_2,\cos\vartheta)Je_m(h_2,\cosh\mu) + \left[\frac{So_m(h_2,\cos\varphi_0)}{M_m^o(h_2)}\right] \cdot So_m(h_2,\cos\vartheta)Jo_m(h_2,\cosh\mu)\right\} \quad (2)$$

In (2) μ and ϑ are the transverse elliptical cylindrical coordinates, $Je_m(Jo_m)$ are the even (odd) radial Mathieu functions of the first kind and $Se_m(So_m)$ are the even (odd) angular Mathieu functions. The normalization constants $M_m^{e(o)}$, are given in [6]. The angle φ_0 defines the direction of incidence of the plane wave with respect to the positive x axis. The assumed time dependence is exp $(j\omega t)$.

The field in region I, expressed in terms of circular cylindrical wave functions has the form

$$E_z^I = \sum_{i=0}^{\infty}[J_i(k_1 r) + q_i N_i(k_1 r)][A_i \cos i\varphi + C_i \sin i\varphi] \quad (3)$$

where r, φ are the polar coordinates, J_i and N_i are the usual cylindrical Bessel functions of the first and second kind, respectively, and

$$q_i = -J_i(k_1 R_1)/N_i(k_1 R_1), \qquad i = 0, 1, 2, \cdots \quad (4)$$

Expression (3) satisfies the boundary condition $E_z^I = 0$ for $r = R_1$. In order to satisfy the boundary conditions at $\mu = \mu_0$, the circular cylindrical wave functions are expanded into concentric elliptical ones by means of well-known formulas [6]:

$$Z_i(kr)\cos i\varphi = \frac{\sqrt{8\pi}}{\epsilon_i}\sum_{m=0}^{\infty}\frac{B_i^e(h,m)}{M_m^e(h)} \cdot Se_m(h, \cos\vartheta) Ze_m(h, \cosh\mu), \quad i = 0, 1, 2\cdots \quad (5)$$

$$Z_i(kr)\sin i\varphi = \frac{\sqrt{8\pi}}{2}\sum_{m=1}^{\infty}\frac{B_i^o(h,m)}{M_m^o(h)} \cdot So_m(h, \cos\vartheta) Zo_m(h, \cosh\mu), \quad i = 1, 2\cdots \quad (6)$$

where i and m are both even or both odd; $Z_i = aJ_i + bN_i$ and $Ze_m = aJe_m + bNe_m(Zo_m = aJo_m + bNo_m)$ are the general Bessel functions and the even (odd) general radial Mathieu functions, respectively; $B_i^{e(o)}(h, m)$ are the well-known expansion coefficients for the Mathieu functions (see the Appendix); and, finally $\epsilon_i = 1$ for $i = 0$ and $\epsilon_i = 2$ for $i \geqq 1$ is the Neumann factor. Substituting from (5), (6) into (3), we find easily the expression for E_z^I in elliptical cylindrical wave functions.

The scattered field, expanded in terms of elliptical cylindrical wave functions, can be written as

$$\begin{aligned} E_z^s = \sum_{m=0}^{\infty}&[P_{2m}Se_{2m}(h_2, \cos\vartheta)He_{2m}^{(2)}(h_2, \cosh\mu) \\ &+ P_{2m+1}Se_{2m+1}(h_2, \cos) \\ &\cdot He_{2m+1}^{(2)}(h_2, \cosh\mu) + Q_{2m}So_{2m}(h_2, \cos\vartheta) \\ &\cdot Ho_{2m}^{(2)}(h_2, \cosh\mu) + Q_{2m+1}So_{2m+1}(h_2, \cos\vartheta) \\ &Ho_{2m+1}^{(2)}(h_2, \cosh\mu)] \end{aligned} \quad (7)$$

where $He_m^{(2)} = Je_m - jNe_m(Ho_m^{(2)} = Jo_m - jNo_m)$.

The total field in region II is $E_z^{\text{II}} = E_z^{\text{inc}} + E_z^s$. Its expression is found using (2) (arranging separately the terms with even and odd m) and (7). So, we can satisfy the boundary condition $E_z^{\text{I}} = E_z^{\text{II}}$ at $\mu = \mu_0$. Multiplying next both members of the resulting equation by $Se_{2m}(h_2, \cos\vartheta)$ and using the orthogonal properties of angular Mathieu functions from [7], we finally find

$$\begin{aligned} &\sum_{i=0}^{\infty}\left\{\frac{\sqrt{8\pi}}{\epsilon_i}A_{2i}\frac{B_{2i}^e(h,2m)}{M_{2m}^e(h)}[Je_{2m}(h, \cosh\mu_0) + q_{2i}Ne_{2m}(h, \cosh\mu_0)]\right\} \\ &\cdot\left[\sum_{n=0}^{\infty}\frac{2\pi}{\epsilon_n}B_{2n}^e(h,2m)B_{2n}^e(h_2,2m)\right] \\ &= \sqrt{8\pi}\, j^{-2m}Se_{2m}(h_2, \cos\varphi_0) \\ &\cdot Je_{2m}(h_2, \cosh\mu_0) + P_{2m}M_{2m}^e(h_2) \\ &\cdot He_{2m}^{(2)}(h_2, \cosh\mu_0) \end{aligned} \quad (8)$$

Repeating the same process multiplying by $Se_{2m+1}, So_{2m}, So_{2m+1}$, respectively, we find other three equations analogous to (8). In the first of them $2i, 2m, 2n$ are substituted by $2i+1, 2m+1, 2n+1$, respectively. In the second, e, A_{2i}, P_{2m} are substituted by o, C_{2i}, Q_{2m}, respectively. Finally, the third is similar to the second, with $2i+1, 2m+1, 2n+1$ in place of $2i, 2m, 2n$, respectively. All three last equations contain 2 in place of ϵ_i and ϵ_n of (8).

The last boundary condition $H_\vartheta^{\text{I}} = H_\vartheta^{\text{II}}$ at $\mu = \mu_0$ is equivalent to $\partial E_z^{\text{I}}/\partial\mu = \partial E_z^{\text{II}}/\partial\mu$ at $\mu = \mu_0$ in this problem, where regions I and II have the same permeability. Satisfying this boundary condition and following steps analogous to those of the previous case, we conclude to four equations similar to (8) and the other three described after it, with the only difference that the various radial Mathieu functions are replaced by their derivatives with respect to μ, at $\mu = \mu_0$.

From these last four equations we can express the scattered field coefficients P_m and Q_m in terms of A's and C's, respectively. So,

$$\begin{aligned} P_m = &\sum_{i=0}^{\infty}\left\{\frac{\sqrt{8\pi}}{\epsilon_i}A_i\frac{B_i^e(h,m)}{M_m^e(h)M_m^e(h_2)} \cdot \left.\frac{\frac{d}{d\mu}[Je_m(h, \cosh\mu) + q_i Ne_m(h, \cosh\mu)]}{\frac{d}{d\mu}He_m^{(2)}(h_2, \cosh\mu)}\right|_{\mu=\mu_0}\right\} \\ &\cdot\left[\sum_{n=0}^{\infty}B_n^e(h,m)B_n^e(h_2,m)\frac{2\pi}{\epsilon_n}\right] - \sqrt{8\pi}\, j^{-m} \\ &\cdot \left.\frac{Se_m(h_2, \cos\varphi_0)}{M_m^e(h_2)}\frac{\frac{d}{d\mu}Je_m(h_2, \cosh\mu)}{\frac{d}{d\mu}He_m^{(2)}(h_2, \cosh\mu)}\right|_{\mu=\mu_0} \\ &(m \geqq 0) \end{aligned} \quad (9)$$

$$Q_m = \sum_{i=0}^{\infty} \left\{ \frac{\sqrt{8\pi}}{2} C_i \frac{B_i^o(h,m)}{M_m^o(h) M_m^o(h_2)} \cdot \left. \frac{\frac{d}{d\mu}[Jo_m(h, \cosh \mu) + q_i No_m(h, \cosh \mu)]}{\frac{d}{d\mu} Ho_m^{(2)}(h_2, \cosh \mu)} \right|_{\mu=\mu_0} \right\} \cdot \left[\sum_{n=0}^{\infty} B_n^o(h,m) B_n^o(h_2, m)\pi \right] - \sqrt{8\pi}\, j^{-m} \cdot \frac{So_m(h_2, \cos\varphi_0)}{M_m^o(h_2)} \left. \frac{\frac{d}{d\mu} Jo_m(h_2, \cosh \mu)}{\frac{d}{d\mu} Ho_m^{(2)}(h_2, \cosh\mu)} \right|_{\mu=\mu_0} \quad (m \geqq 1) \tag{10}$$

where m, i, n are all three even or all three odd.

Substituting then the values of P_m, Q_m from (9), (10) into (8) and the three analogous equations described after (8), we find the following four infinite sets of linear nonhomogeneous equations for the expansion coefficients $A_{2i}, A_{2i+1}, C_{2i}, C_{2i+1}$:

$$\sum_{i=0}^{\infty} a_{2m,2i} A_{2i} = b_{2m},$$

$$\sum_{i=0}^{\infty} a_{2m+1,2i+1} A_{2i+1} = b_{2m+1}, \qquad (m \geqq 0) \tag{11}$$

$$\sum_{i=0}^{\infty} g_{2m,2i} C_{2i} = d_{2m}, \qquad (m > 0),$$

$$\sum_{i=0}^{\infty} g_{2m+1,2i+1} C_{2i+1} = d_{2m+1}, \qquad (m \geqq 0) \tag{12}$$

where

$$a_{mi} = \frac{1}{\epsilon_i} \frac{B_i^e(h,m)}{M_m^e(h)} \left[\sum_{n=0}^{\infty} \frac{2\pi}{\epsilon_n} B_n^e(h,m) B_n^e(h_2, m) \right] \cdot \left\{ [Je_m(h, \cosh\mu_0) + q_i Ne_m(h, \cosh\mu_0)] - He_m^{(2)}(h_2, \cosh\mu_0) \cdot \left. \frac{\frac{d}{d\mu}[Je_m(h, \cosh \mu) + q_i Ne_m(h, \cosh \mu)]}{\frac{d}{d\mu} He_m^{(2)}(h_2, \cosh\mu)} \right|_{\mu=\mu_0} \right\} \tag{13}$$

$$b_m = j^{-m} Se_m(h_2, \cos\varphi_0) \left[Je_m(h_2, \cosh\mu_0) - He_m^{(2)}(h_2, \cosh\mu_0) \left. \frac{\frac{d}{d\mu} Je_m(h_2, \cosh\mu)}{\frac{d}{d\mu} He_m^{(2)}(h_2, \cosh\mu)} \right|_{\mu=\mu_0} \right] \tag{14}$$

$$g_{mi} = \frac{1}{2} \frac{B_i^o(h,m)}{M_m^o(h)} \left[\sum_{n=1}^{\infty} \pi B_n^o(h,m) B_n^o(h_2, m) \right] \cdot \left\{ [Jo_m(h, \cosh\mu_0) + q_i No_m(h, \cosh\mu_0)] - Ho_m^{(2)}(h_2, \cosh\mu_0) \cdot \left. \frac{\frac{d}{d\mu}[Jo_m(h, \cosh \mu) + q_i No_m(h, \cosh \mu)]}{\frac{d}{d\mu} Ho_m^{(2)}(h_2, \cosh\mu)} \right|_{\mu=\mu_0} \right\} \tag{15}$$

$$d_m = j^{-m} So_m(h_2, \cos\varphi_0) \left[Jo_m(h_2, \cosh\mu_0) - Ho_m^{(2)}(h_2, \cosh\mu_0) \left. \frac{\frac{d}{d\mu} Jo_m(h_2, \cosh\mu)}{\frac{d}{d\mu} Ho_m^{(2)}(h_2, \cosh\mu)} \right|_{\mu=\mu_0} \right] \tag{16}$$

and m, i, n are all three even or all three odd.

For general values of $h (h_2 = \tau h, \tau = k_2/k_1 = \sqrt{\epsilon_2/\epsilon_1})$, the infinite sets given by (11), (12) can be solved by numerical methods only. The complications of such an approach are discussed in [1], [2]. For small values of h an analytical solution is, however, possible. After long, very laborious calculations one can find expansions of the following form:

$$a_{mm}(h) = A_{mm} + \mathcal{A}_{mm} h^2 + O(h^4),$$

$$g_{mm}(h) = G_{mm} + \mathcal{G}_{mm} h^2 + O(h^4) \tag{17}$$

$$a_{mi}(h) = A'_{mi} h^{|i-m|} [1 + O(h^2)],$$

$$g_{mi}(h) = G'_{mi} h^{|i-m|} [1 + O(h^2)], \quad (i \neq m) \tag{18}$$

$$b_m(h) = B_m + \mathcal{B}_m h^2 + O(h^4),$$

$$d_m(h) = D_m + \mathcal{D}_m h^2 + O(h^4). \tag{19}$$

But here m and i are both even or both odd, so a_{mi} and $g_{mi} (i \neq m)$ are the order of h^2 or higher.

The analytical expressions of the expansion coefficients A, B, G and D calculated from (13)–(16) are given in the Appendix. A simple comparison of the expressions (13)–(16) with the corresponding ones of [5] shows the much more complexity of the present expressions and calculations.

The systems (11) and (12) have exactly the same forms with those of [5]. So the solution for A_i taken from [5] is

$$A_i = \frac{1}{a_{ii}} \left(-\delta_i b_{i-2} \frac{a_{i,i-2}}{a_{i-2,i-2}} + b_i - b_{i+2} \frac{a_{i,i+2}}{a_{i+2,i+2}} \right), \quad i = 0, 1, 2, \cdots \tag{20}$$

where C_i is found from (20) if a, b, δ_i are substituted by $g, d, \delta_{i-1} (i \geqq 1$, respectively. In (20) $\delta_0 = \delta_1 = 0$ and $\delta_i = 1$ for $i \geqq 2$, since m and i are never negative in (11), (12).

III. The Scattered Field

The scattered field is given in (7). Using formula (6) from [5] and the similar for odd functions, we can express this field into circular cylindrical wave functions. Using then the asymptotic expansion for Hankel functions [5] we take the scattered far-field expression

$$E_z^s = \frac{e^{-jk_2r}}{\sqrt{r}} f(\varphi) \tag{21}$$

in terms of the scattering amplitude

$$f(\varphi) = \frac{1+j}{\sqrt{2k_2}} G(\varphi) \tag{22}$$

and

$$G(\varphi) = \sum_{m=0}^{\infty}\sum_{l=0}^{\infty} j^l [P_m B_l^e(h_2, m)\cos l\varphi + Q_m B_l^o(h_2, m)\sin l\varphi]. \tag{23}$$

In (23) m and l are both even or both odd.

The differential, backscattering (radar), forward and total scattering cross sections are, respectively:

$$\sigma(\varphi) = |f(\varphi)|^2 \tag{24}$$

$$\sigma_b = 2\pi\sigma(\varphi_0 + \pi) = 2\pi|f(\varphi_0+\pi)|^2 \quad \text{or} \quad k_2\sigma_b = 2\pi|G(\varphi_0+\pi)|^2 \tag{25}$$

$$\sigma_f = 2\pi\sigma(\varphi_0) = 2\pi|f(\varphi_0)|^2 \quad \text{or} \quad k_2\sigma_f = 2\pi|G(\varphi_0)|^2 \tag{26}$$

$$Q = \int_0^{2\pi} \sigma(\varphi)\,d\varphi = \int_0^{2\pi} |f(\varphi)|^2\,d\varphi. \tag{27}$$

Substituting from (22) and (23) in (27) and using the orthogonality relations between trigonometric functions and formulas (A29) from the Appendix, we finally find after very lengthy calculations the result:

$$k_2Q_t = 2\pi|P_0|^2[B_o^e(h_2,0)]^2 + \pi\sum_{m=1}^{\infty}\{|P_m|^2[B_m^e(h_2,m)]^2 + |Q_m|^2[B_m^o(h_2,m)]^2\}. \tag{28}$$

In (28) we have kept terms up to the order h^2, only. The coefficients P_m and Q_m are given in (9) and (10), respectively.

Using (17)–(19) in (20) and the analogous for C_i one finds that

$$A_i(h) = A_i^o + A_i^{(2)}h^2 + O(h^4),$$
$$C_i(h) = C_i^o + C_i^{(2)}h^2 + O(h^4). \tag{29}$$

The various expansion coefficients in (29) are given in the Appendix.

Substituting from (29) into (9) and (10) and using formulas for Mathieu functions from [5], [6], as well as (A25)–(A30), from the Appendix, we obtain after lengthy manipulation analogous expansions for P_m and Q_m and finally for $\sigma(\varphi)$ and Q_t. The last two can be set in the form

$$S(h) = S(0) + Sh^2 + O(h^4) = S(0)[1 + gh^2 + O(h^4)], \qquad g = \frac{S}{S(0)} \tag{30}$$

where both g are independent of h, while that for Q_t is also independent of φ. The expansions for σ_b and σ_f are calculated from (30) for special values of φ. The expansion coefficients $S(0)$ are the well-known results for the problem of two coaxial circular cylinders, while g were found after very laborious and lengthy calculations and it is impossible to give their analytical expressions, without making the paper very long.

Our results were checked by the use of the forward scattering theorem [8], which in this case has the form:

$$Q_t = -2(\pi/k_2)^{1/2}\,\mathrm{Re}\,[(1-j)f(\varphi_0)] = -[2(2\pi)^{1/2}/k_2]\,\mathrm{Re}\,[G(\varphi_0)]. \tag{31}$$

The very lengthy expressions of the various coefficients do not permit the analytical proof of (31), but its validity was established numerically for many values of the parameters.

IV. The H-Wave Polarization

In this section the case of the H-wave polarization is examined. The incident wave H_z^{inc} impinging normally on the z axis is expressed by formula (2). The expansion for H_z^{I} in region I is given by (3) in circular cylindrical wave functions and with the use of (5), (6) into (3), in elliptical cylindrical wave functions, with the only difference that q_i in this case has the expression

$$q_i = -J_i'(k_1R_1)/N_i'(k_1R_1), \qquad i = 0,1,2,\cdots \tag{32}$$

in order to satisfy the boundary condition $\partial H_z^{\mathrm{I}}/\partial r = 0$ (corresponding to $E_\varphi^{\mathrm{I}} = 0$) at $r = R_1$. In (32) the primes denote derivatives of Bessel functions with respect to the argument. The scattered field H_z^s is given by (7). The total field in region II is $H_z^{\mathrm{II}} = H_z^{\mathrm{inc}} + H_z^s$. Satisfying the boundary conditions at $\mu = \mu_0$:

$$H_z^{\mathrm{I}} = H_z^{\mathrm{II}}, \quad (\epsilon_2/\epsilon_1)\partial H_z^{\mathrm{I}}/\partial\mu = \partial H_z^{\mathrm{II}}/\partial\mu \tag{33}$$

(the second corresponding to $E_\vartheta^{\mathrm{I}} = E_\vartheta^{\mathrm{II}}$) and following steps identical to those for the case of the E wave we find equations similar to (8)–(31). The only differences here are that the first infinite sums ($\Sigma_{i=0(1)}^{\infty}$) in (9) and (10) are multiplied by ϵ_2/ϵ_1, while their last terms remain unchanged and that the last terms in the right parentheses of (13) and (15). i.e., those multiplied by $He_m^{(2)}$ and $Ho_m^{(2)}$, respectively, are also multiplied by ϵ_2/ϵ_1. The formulas (14) and (16) remain the same, also in this case. It should be noticed that the expression for q_i used in the various equations of the present case, is given in (32).

V. Numerical Results and Discussion

In Figs. 2–9 the various scattering cross sections are given for the configuration of Fig. 1, for the cases $h = 0$ (problem of two coaxial circular cylinders) and $h = 0.05, 0.1$, for various values of the parameters. The expansion coefficients g are determined from expressions (30). The physical restriction $a/2 = (R_2^2 - r_2^2)^{1/2} \leqq R_2[1 - (R_1/R_2)^2]^{1/2}$ imposes an upper limit on the values of $h = ka/2$, so that the restriction $h \ll 1$ is not so severe as may appear at first.

The terms omitted in all of our results are of order h^4 and greater and so h can take relatively large values in our solution.

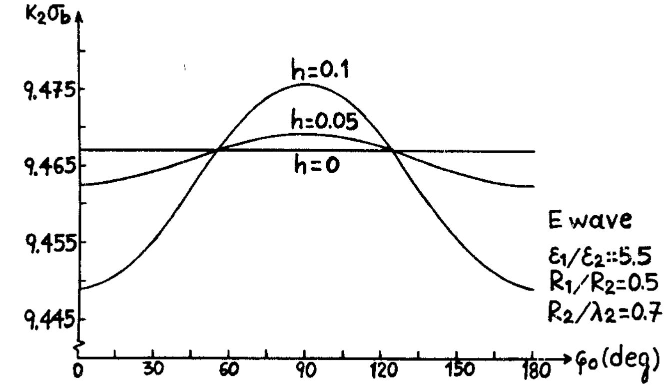

Fig. 2. Backscattering cross section for $\epsilon_1/\epsilon_2 = 5.5, R_2/\lambda_2 = 0.7$ (E wave).

In Figs. 2–5 the backscattering (radar) cross section σ_b is given for various values of the parameters and for both polarizations. The same is done in Figs. 6 and 7 for the forward scattering cross section σ_f and in Figs. 8 and 9 for the total scattering cross section Q_t. In all these figures $\lambda_2 = 2\pi/k_2$ is the wavelength in region II (free space). For certain values of the parameters it is obvious that by making the outer dielectric cylinder elliptic, one can produce a larger or smaller σ_b, σ_f or Q_t, in comparison with that of the problem of two coaxial circular cylinders. Analogous results have also been found in [3]–[5]. This may be useful for certain applications.

All of our results are symmetrical about the x and y axes, as is imposed by the geometry of the scatterer.

APPENDIX

The various expansion coefficients appearing in (17)–(19) are given below: In the above relations we have made the substitutions: In all the above formulas $H_i = H_i^{(2)}(x\ )$ is the Hankel function of the second kind and H_t' its derivative with respect to the argument.

Using next (A1)–(A6) in (20) we find the expressions for

$$A_{mm} = \frac{1}{\epsilon_m}\sqrt{\frac{\pi}{2}}\left(I_{mm} - \frac{K_{mmm}}{\tau H'_m}\right), \qquad (m \geqq 0) \tag{A1}$$

$$\begin{aligned}A_{mm} = \frac{1}{\epsilon_m}\sqrt{\frac{\pi}{2}}\Bigg\{&-\delta_m\frac{I_{m-2,m}}{r_m} - \frac{(\tau^2+1)I_{mm}}{s_m} + \frac{I_{m+2,m}}{t_m} + \Bigg[\Bigg(\delta_m\frac{X_m^-}{r_m} + \frac{2\tau^2+1}{s_m}K_{mmm}\\ &- \frac{X_m^+}{t_m}\Bigg)H'_m + K_{mmm}\tau^2\Bigg(-\delta_m\frac{H'_{m-2}}{r_m} - \frac{H'_m}{s_m} + \frac{H'_{m+2}}{t_m}\Bigg)\Bigg]\Bigg/(\tau H'^2_m)\Bigg\}, \qquad (m \geqq 0)\end{aligned} \tag{A2}$$

$$A'_{m,m-2} = \frac{1}{\epsilon_{m-2}}\sqrt{\frac{\pi}{2}}\delta_m\left(I_{m,m-2} - \frac{K_{m,m,m-2}}{\tau H'_m}\right),\Big/ r_m \qquad (m \geqq 0) \tag{A3}$$

$$A_{m,m+2} = -\frac{1}{2}\sqrt{\frac{\pi}{2}}\left(I_{m,m+2} - \frac{K_{m,m,m+2}}{\tau H'_m}\right),\Big/ t_m \qquad (m \geqq 0) \tag{A4}$$

$$B_m = j^{-m}\sqrt{\frac{\pi}{2}}\left(L_{mm} - \frac{F_{mmm}}{H'_m}\right), \qquad (m \geqq 0) \tag{A5}$$

$$\begin{aligned}B_m = j^{-m}\sqrt{\frac{\pi}{2}}\tau^2\Bigg\{&\delta_m\frac{L_m^-}{r_m} - \frac{2L_{mm}}{s_m} - \frac{L_m^+}{t_m} - \Bigg[\Bigg(\delta_m\frac{F_m^-}{r_m} - \frac{3F_{mmm}}{s_m} - \frac{F_m^+}{t_m}\Bigg)H'_m\\ &+ F_{mmm}\Bigg(\delta_m\frac{H'_{m-2}}{r_m} + \frac{H'_m}{s_m} - \frac{H'_{m+2}}{t_m}\Bigg)\Bigg]\Bigg/H'^2_m\Bigg\}, \qquad (m \geqq 0)\end{aligned} \tag{A6}$$

$$G_{mm} = \frac{1}{2m}\sqrt{\frac{\pi}{2}}\tanh\,\mu_0\left(I_{mm} - \frac{K_{mmm}}{\tau H'_m}\right), \qquad (m \geqq 0) \tag{A7}$$

$$\begin{aligned}G_{mm} = \frac{1}{2m}\sqrt{\frac{\pi}{2}}\tanh\,\mu_0\Bigg[&-\delta_{m-1}\frac{m-2}{m}\frac{I_{m-2,m}}{r_m} + \frac{(\tau^2+1)I_{mm}}{s_m} + \frac{m+2}{m}\frac{I_{m+2,m}}{t_m}\\ &+ \frac{H'_m\Delta_m + K_{mmm}E_m}{\tau(x_3H'_m)^2}\Bigg] \qquad (m \geqq 1)\end{aligned} \tag{A8}$$

$$G'_{m,m-2} = \frac{1}{2m}\sqrt{\frac{\pi}{2}}\tanh\,\mu_0\delta_{m-1}\left(I_{m,m-2} - \frac{K_{m,m,m-2}}{\tau H'_m}\right),\Big/ r_m \qquad (m \geqq 1) \tag{A9}$$

$$G'_{m,m+2} = -\frac{1}{2m}\sqrt{\frac{\pi}{2}}\tanh\,\mu_0\left(I_{m,m+2} - \frac{K_{m,m,m+2}}{\tau H'_m}\right),\Big/ t_m \qquad (m \geqq 1) \tag{A10}$$

$$D_m = \frac{j^{-m}}{m}\sqrt{\frac{\pi}{2}}\tanh\,\mu_0\left(\Phi_{mm} - \frac{S_{mmm}}{H'_m}\right), \qquad (m \geqq 1) \tag{A11}$$

$$\begin{aligned}D_m = \frac{j^{-m}}{m}\sqrt{\frac{\pi}{2}}\tanh\,\mu_0\Bigg[&\delta_{m-1}\frac{\Phi_{m-2,m-((m-2)/m)}\Phi_{m,m-2}}{r_m} + \frac{2\Phi_{mm}}{s_m}\\ &- \frac{\Phi_{m+2,m} - ((m+2)/m)\Phi_{m,m+2}}{t_m} - \frac{H'_mU_m - S_{mmm}E_m}{(x_2H'_m)^2}\Bigg]\tau^2, \qquad (m \geqq 1).\end{aligned} \tag{A12}$$

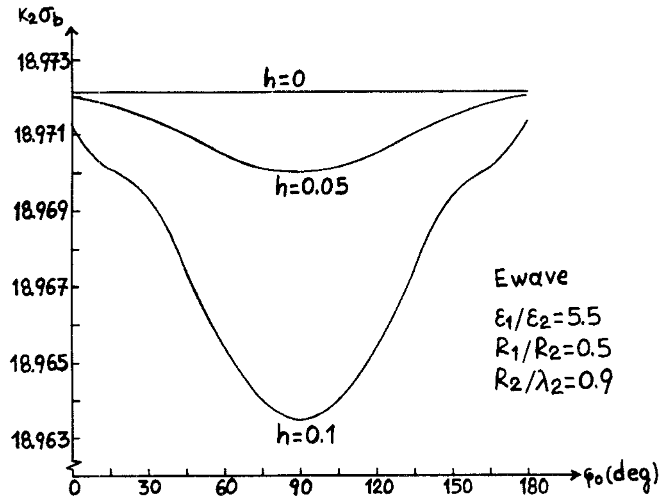

Fig. 3. Backscattering cross section for $\epsilon_1/\epsilon_2 = 5.5, R_2/\lambda_2 = 0.9$ (E wave).

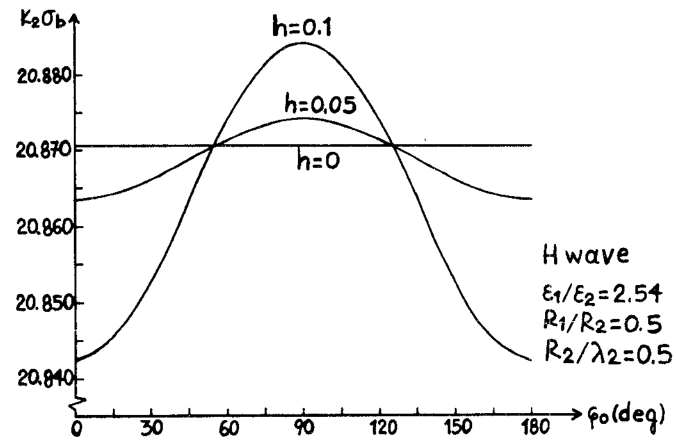

Fig. 4. Backscattering cross section for $\epsilon_1/\epsilon_2 = 2.54, R_2/\lambda_2 = 0.5$ (H wave).

the coefficients A_i^o and $A_i^{(2)}$, appearing in (29):

$$A_i^o = B_i/A_{ii}, \quad A_i^{(2)} = \frac{A_{ii}B_i - B_iA_{ii}}{A_{ii}^2} - \delta_i\frac{B_{i-2}A'_{i,i-2}}{A_{i-2,i-2}A_{ii}} - \frac{B_{i+2}A'_{i,i+2}}{A_{ii}A_{i+2,i+2}}. \quad (i \geqq 0) \tag{A24}$$

The coefficients C_i^o and $C_i^{(2)}$ are given by (A24) if we use G, D (from (A7)–(A12)), $\delta_{i-1}(i \geqq 1)$ in place of A, B, δ_i, respectively.

In order to calculate all the preceding expansion coefficients, we have expanded the coefficients $B_m^{e(o)}(h, i)$ for the Mathieu functions, into powers of h, following procedures outlined in [6] for small values of h. The calculations, straightforward but very laborious, were made up to order h^2, for many values of $m(i = m, m \pm 2)$ and we have found the following general formulas, valid for each value of m, with only two exceptions referred below:

$$B_{m\pm2}^e(h,m) = \mp\frac{h^2}{8\epsilon_m(m\pm1)} + O(h^4), \quad \left(m \geqq \begin{matrix} 0 & \text{(upper sign)} \\ 2 & \text{(lower sign)} \end{matrix}\right) \tag{A25}$$

$$B_m^e(h,m) = 1 - \frac{h^2}{8(m^2-1)} + O(h^4), \qquad (m \neq 1) \tag{A26}$$

$$I_{ln} = J_l(x_3) - (J_n(x_1)/N_n(x_1))N_l(x_3),$$

$$I'_{ln} = J'_l(x_3) - (J_n(x_1)/N_n(x_1))N'_l(x_3), \tag{A13}$$

$$x_1 = k_1R_1, \quad x_2 = k_2R_2, \quad x_3 = k_1R_2 \tag{A14}$$

$$K_{iln} = H_iI'_{ln}, \quad X_m^{\pm} = K_{m,m\pm2,m} + \tau^2K_{m\pm2,m,m} \tag{A15}$$

$$r_m = 16(m-1), \quad t_m = 8\epsilon_m(m+1) \tag{A16}$$

$$s_m = 8(m^2-1), \qquad (m \pm 1), \quad s_1 = \begin{cases} -32 & (\text{for } A, B) \\ 32/3 & (\text{for } G, D) \end{cases} \tag{A17}$$

$$L_{ln} = J_l(x_2)\cos(n\varphi_0), \quad F_{iln} = H_iJ'_l(x_2)\cos(n\varphi_0) \tag{A18}$$

$$L_m^{\pm} = L_{m,m\pm2} - L_{m\pm2,m}, \quad F_m^{\pm} = F_{m,m,m\pm2} - F_{m,m\pm2,m} - F_{m\pm2,m,m} \tag{A19}$$

$$\Delta_m = \left(\delta_{m-1}\frac{m-2}{m}\frac{K_{m,m-2,m} + \tau^2K_{m-2,m,m}}{r_m} - \frac{(2\tau^2+1)K_{mmm}}{s_m} - \frac{m+2}{m}\frac{K_{m,m+2,m} + \tau^2K_{m+2,m,m}}{t_m}\right)x_3^2 - \frac{H_mI_{mm}}{x_3} + K_{mmm} \tag{A20}$$

$$E_m = \left(-\delta_{m-1}\frac{m-2}{m}\frac{H'_{m-2}}{r_m} + \frac{H'_m}{s_m} + \frac{m+2}{m}\frac{h'_{m+2}}{t_m}\right)x_2^2 + \frac{H_m}{x_2} - H'_m \tag{A21}$$

$$\Phi_{ln} = \sin(l\varphi_0)J_n(x_2), \quad S_{iln} = \sin(i\varphi_0)H_lJ'_n(x_2) \tag{A22}$$

$$U_m = \left\{\frac{\delta_{m-1}[S_{m-2,m,m} - ((m-2)/m)(S_{m,m-2,m} + S_{m,m,m-2})]}{r_m} + \frac{3S_{mmm}}{s_m} - \frac{S_{m+2,m,m} - ((m+2)/m)(S_{m,m,m+2} + S_{m,m+2,m})}{t_m}\right\}x_2^2 + \frac{H_m\Phi_{mm}}{x_2} - S_{mmm}. \tag{A23}$$

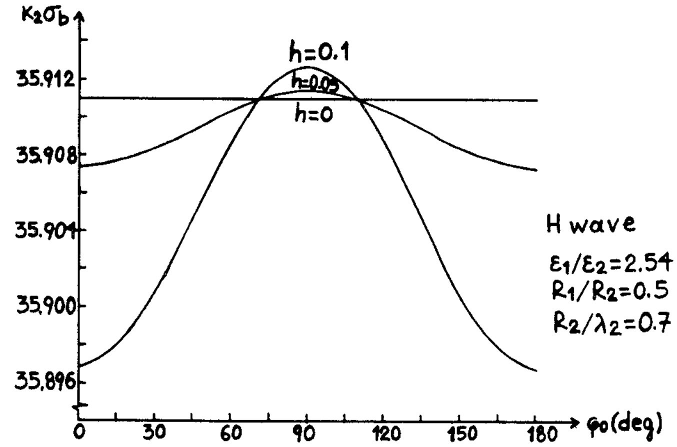

Fig. 5. Backscattering cross section for $\epsilon_1/\epsilon_2 = 2.54, R_2/\lambda_2 = 0.7$ (H wave).

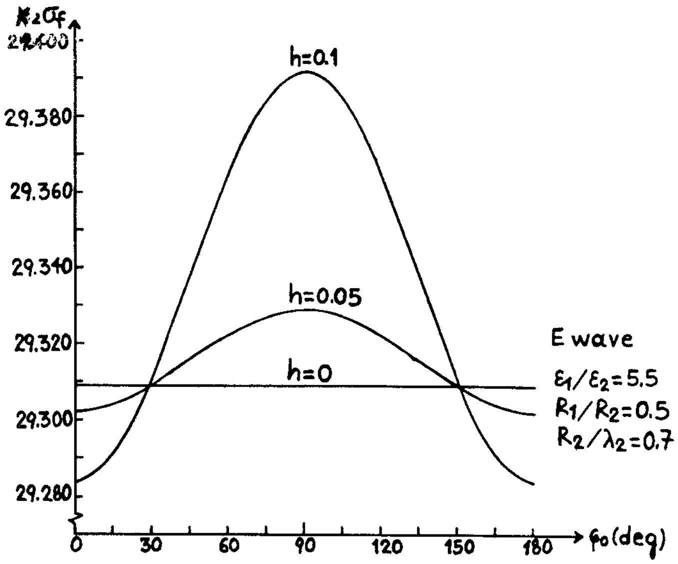

Fig. 6. Forward scattering cross section for $\epsilon_1/\epsilon_2 = 5.5, R_2/\lambda_2 = 0.7$ (E wave).

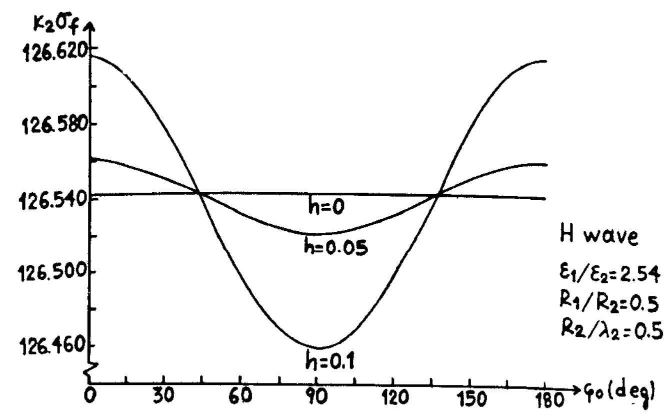

Fig. 7. Forward scattering cross section for $\epsilon_1/\epsilon_2 = 2.54, R_2/\lambda_2 = 0.5$ (H wave).

$$B^o_{m\pm 2}(h,m) = \mp\frac{h^2}{16m(m\pm 1)} + O(h^4), \quad \left(m \geqq \begin{matrix} 1 & \text{(upper sign)} \\ 3 & \text{(lower sign)} \end{matrix}\right) \tag{A27}$$

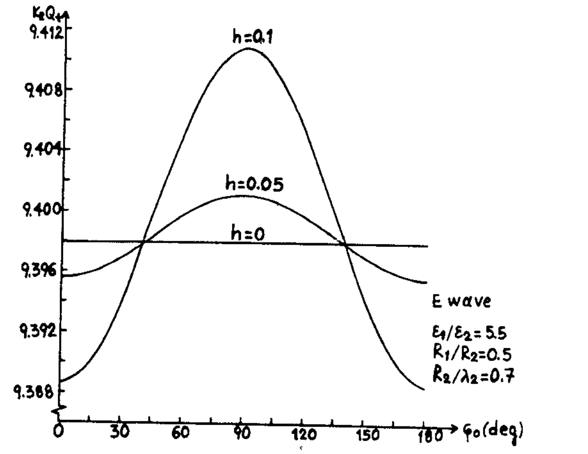

Fig. 8. Total scattering cross section for $\epsilon_1/\epsilon_2 = 5.5, R_2/\lambda_2 = 0.7$ (E wave).

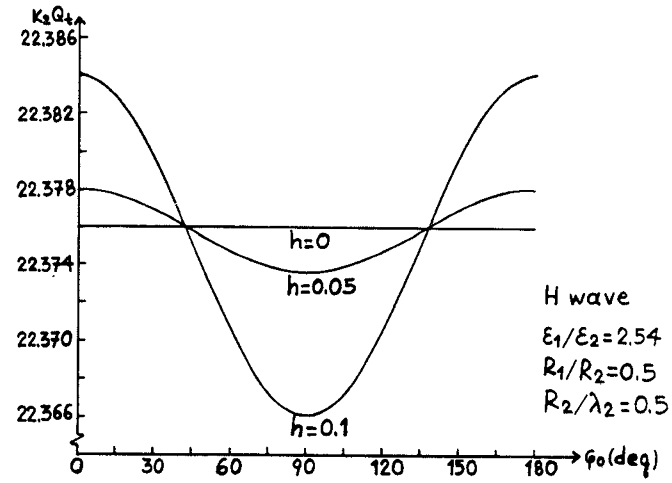

Fig. 9. Total scattering cross section for $\epsilon_1/\epsilon_2 = 2.54, R_2/\lambda_2 = 0.5$ (H wave).

$$B^o_m(h,m) = \frac{1}{m} + \frac{h^2}{8m(m^2-1)} + O(h^4), \qquad (m \geqq 2) \tag{A28}$$

From (A25) and (A27) there results that up to order h^2:

$$\begin{aligned} \epsilon_m B^e_{m-2}(h,m) &= -\epsilon_{m-2} B^e_m(h,m-2), \\ &\qquad (m \geqq 2) \\ B^o_{m-2}(h,m)/(m-2) &= -B^o_m(h,m-2)/m, \\ &\qquad (m > 2) \end{aligned} \tag{A29}$$

The two coefficients not given by (A25)–(A28) are

$$\begin{aligned} B^e_1(h,1) &= 1 + \frac{h^2}{32} + O(h^4), \\ B^o_1(h,1) &= 1 + \frac{3h^2}{32} + O(h^4). \end{aligned} \tag{A30}$$

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